# Fast basecases for arbitrary-size multiplication

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# Fast multiple-precision arithmetic – why?

We care about fast multiple-precision arithmetic. *Why?*

Examples include:

- Correctly rounded floating point arithmetic without the use of lookup tables (e.g. MPFR [1])
- Verifying the Riemann hypothesis up to very big numbers [2]
- Certified homotopy continuation (ex. [3])

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# Basics of Multiple-Precision Arithmetic

Fundamentals are these schoolbook $\mathcal{O}(n)$ operations:

- Left and right shift: $r \leftarrow \lfloor a \cdot 2^e \rfloor$
- Addition and subtraction: $r \leftarrow a \pm b$
- $m \times 1$-multiplication: $r \leftarrow a \cdot b_0$ (`mul_1`)
- Addition of $m \times 1$-multiplication: $r \leftarrow r + a \cdot b_0$ (`addmul_1`)

Schoolbook $m \times n$-multiplication is then

$r \leftarrow a \cdot b_0$ // `mul_1`
**for** $i \leftarrow 1$ *to* $n-1$ **do**
  $r \leftarrow r + (a \cdot b_i) \cdot \beta^i$ // `addmul_1`
**end**

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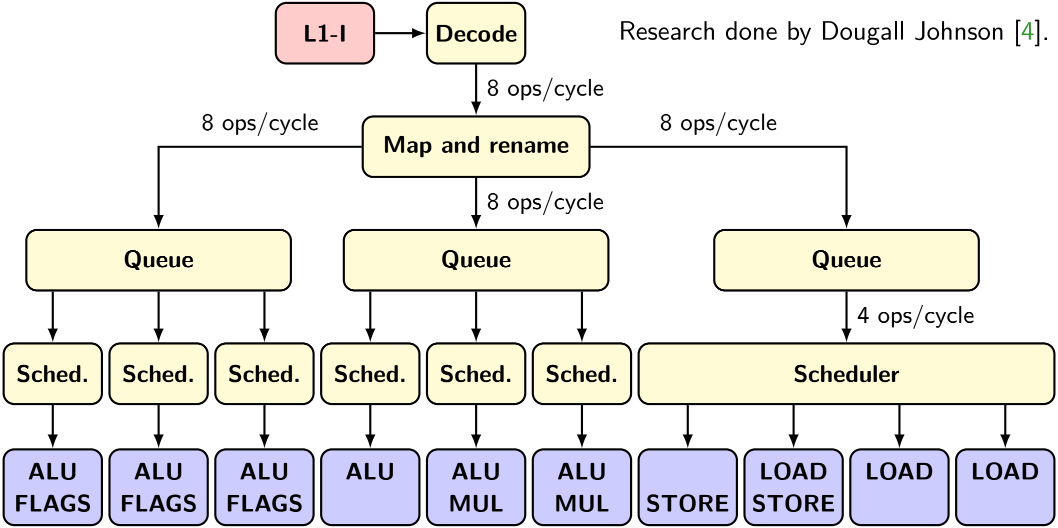
# Apple M1 Pipeline (Simplified)

Research done by Dougall Johnson [4].



# CPU Pipeline

Simple version:

1. Read some instructions from memory
2. Schedule the instructions to the correct unit
3. Units executes instructions

This scheme allows for:

- Concurrent execution of multiple instructions
- Out-of-order execution

But one has to be aware of dependency chains.

**Example:** (Dependency chain)
Consider the algorithm

$$x_0 \leftarrow a_0\,,$$
$$x_1 \leftarrow x_0 + a_1\,,$$
$$x_2 \leftarrow x_1 + a_2\,.$$

$x_0$ needs to be evaluated before $x_1$ can be computed. And $x_1$ needs to be evaluated before $x_2$ can be computed. This is called a *dependency chain*.

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# Lower bound of GMP's `addmul_1`

```
L(top): ldp     u0, u1, [up], #16        adds    x0, r0, x0
        ldp     u2, u3, [up], #16        adcs    u0, r1, u0
        ldp     r0, r1, [rp]             adcs    u1, r2, u1
        ldp     r2, r3, [rp,#16]         adcs    u2, r3, u2
        mul     x0, u0, v0               adc     u3, u3, zero
        umulh   u0, u0, v0               adds    x0, x0, CY
        mul     x1, u1, v0               adcs    u0, u0, x1
        umulh   u1, u1, v0               adcs    u1, u1, x2
        mul     x2, u2, v0               adcs    u2, u2, x3
        umulh   u2, u2, v0               adc     CY, u3, zero
        mul     x3, u3, v0               stp     x0, u0, [rp], #16
        umulh   u3, u3, v0               stp     u1, u2, [rp], #16
                                         sub     n, n, #1
                                         cbnz    n, L(top)
```

| **Unit type (amount)** | **Cycles / 4 words** |
|---|---|
| LOAD/STORE (3/2) | |
| MUL (2) | |
| ALU+FLAGS (3) | |

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| ALU+FLAGS (3) | |

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| MUL (2) | 4 |
| ALU+FLAGS (3) | ~~4~~ 5 |

*5 cycles per 4 words?*

**Benchmarks says yes!**

# What can be improved?

On Apple M1, GMP's `addmul_1` will do $\frac{k+1}{k}$ cycles per word *asymptotically*, where $k$ is the number of unrolls. Can we improve its performance?

The idea is to fully unroll one size parameter in the multiplication, and have a lookup table for all the basecase sizes. This will:

- Reduce some overhead,
- Avoid breaking carry chains, hopefully let

$$\frac{n+1}{n} \text{ cycles per word} \longrightarrow 1 \text{ cycle per word}$$

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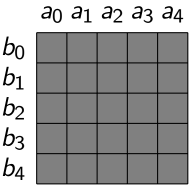
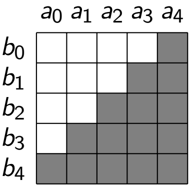
# High multiplication

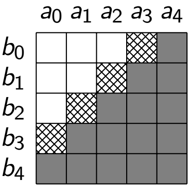
High multiplication is multiplication where we scrap the lower part of the result. Important use cases are floating point arithmetic and modular arithmetic.



Full multiplication



Sloppy approximate



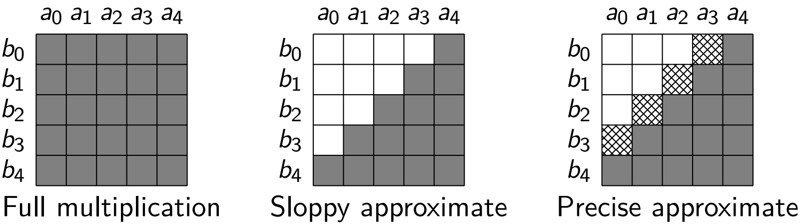
Precise approximate

☐ – scrapped

▨ – high multiplication between two words $u$ and $v$: $\lfloor uv/\beta \rfloor$

■ – full multiplication

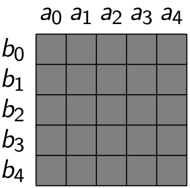
# High multiplication (cont.)




- Full multiplication: $n^2$ multiplications, no error
- Sloppy approximate: $\frac{n\cdot(n+1)}{2}$ multiplications, error $< (n-1)\beta^n$
- Precise approximate: $\frac{n\cdot(n+1)}{2}$ full and $n-1$ high multiplications, error $< (2n-3)\beta^{n-1}$

With precise approximate we can check if the upper $n$ words are guaranteed to be correctly rounded.

# High multiplication (cont.)

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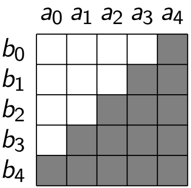


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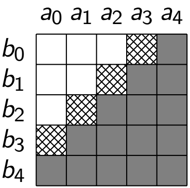


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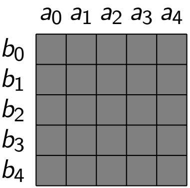

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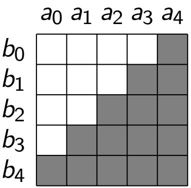

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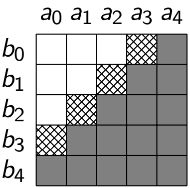

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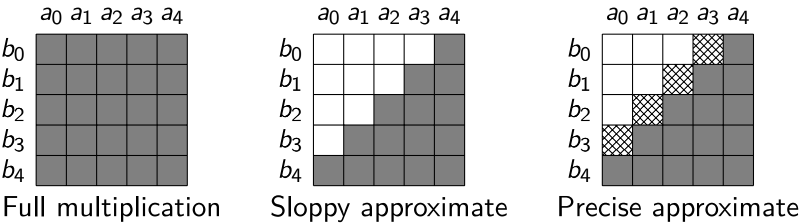


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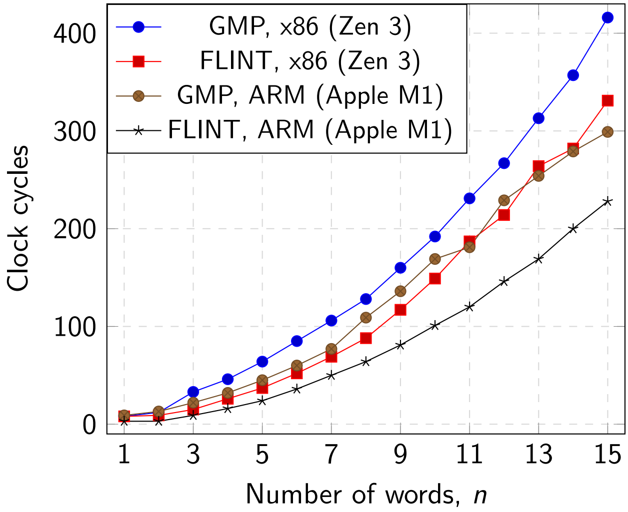
# Results, full multiplication



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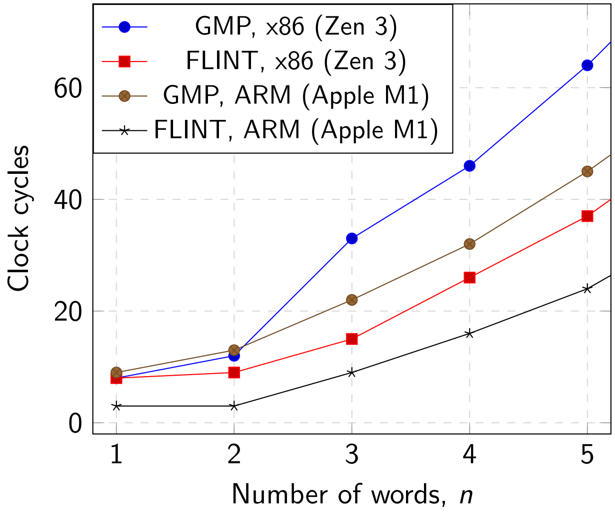

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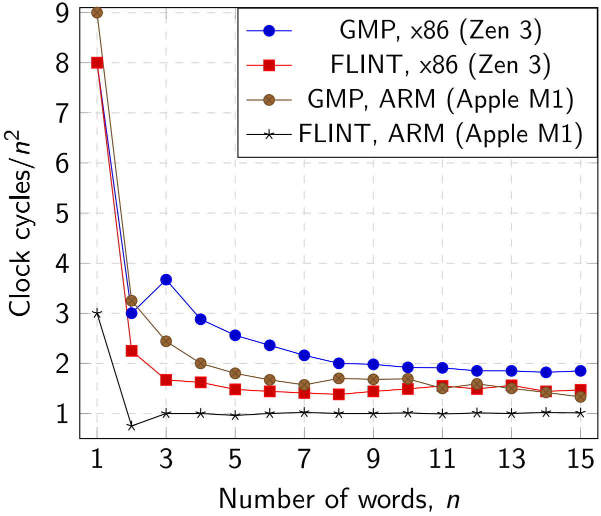
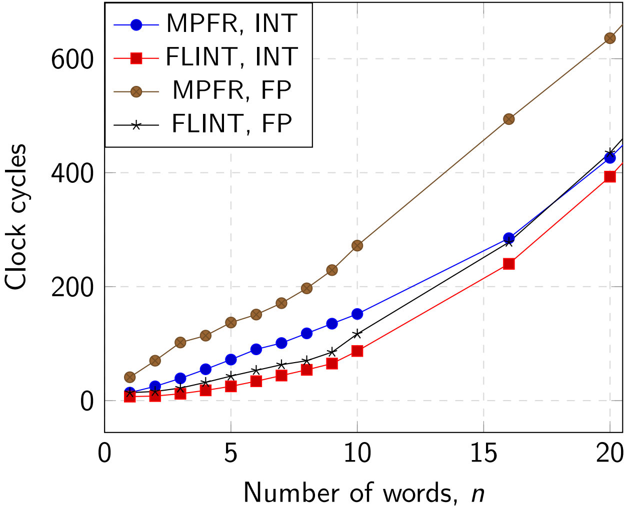


# $10^7$ multiplications with lengths $m, n \in \{1, 2, \ldots, N\}$, uniformly random

| | GMP (`mpn_mul`) | | | | Ours (`flint_mpn_mul`) | | | |
|---|---|---|---|---|---|---|---|---|
| $N$ | Time | C | J | I | Time | C | J | I |
| | Random, x86-64 (Zen 3) | | | | | | | |
| 8 | 0.32 s | 18.3% | 22.3% | 0% | 0.18 s | 20.9% | 48.4% | 0.00% |
| 16 | 0.55 s | 10.0% | 18.2% | 0% | 0.43 s | 14.3% | 33.1% | 3.05% |
| 32 | 1.39 s | 10.5% | 12.7% | 0% | 1.32 s | 10.7% | 16.7% | 0.41% |
| 64 | 4.48 s | 11.5% | 11.6% | 0% | 4.29 s | 12.9% | 14.3% | 0.12% |
| | Random, ARM64 (M1) | | | | | | | |
| 8 | 0.30 s | 11.4% | 0.00% | 0.00% | 0.23 s | 11.2% | 41.7% | 0.01% |
| 16 | 0.50 s | 10.9% | 0.00% | 0.00% | 0.43 s | 10.5% | 41.6% | 0.00% |
| 32 | 1.31 s | 9.6% | 0.00% | 0.00% | 1.13 s | 10.0% | 13.9% | 0.00% |
| 64 | 4.16 s | 8.3% | 0.20% | 0.02% | 3.82 s | 9.8% | 4.2% | 0.06% |

Table: Conditional branch misprediction rates “C”, indirect jump address misprediction rates “J” and instruction cache miss rates “I”.

# Results, high multiplication on Zen 3



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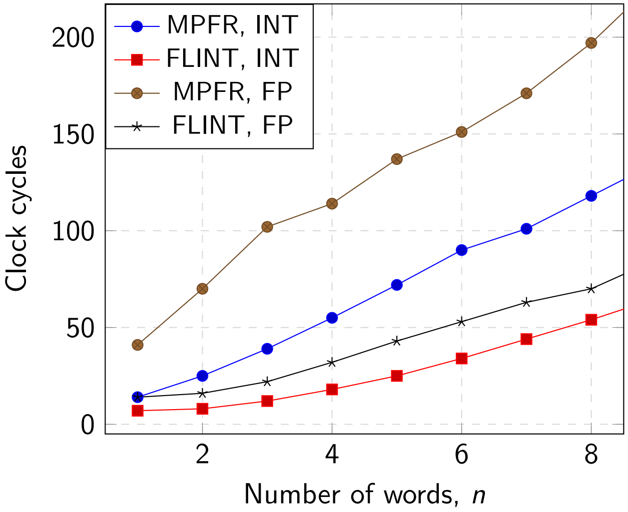

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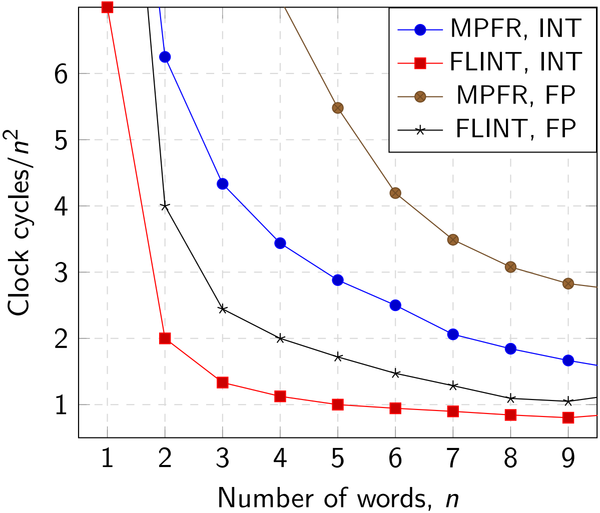
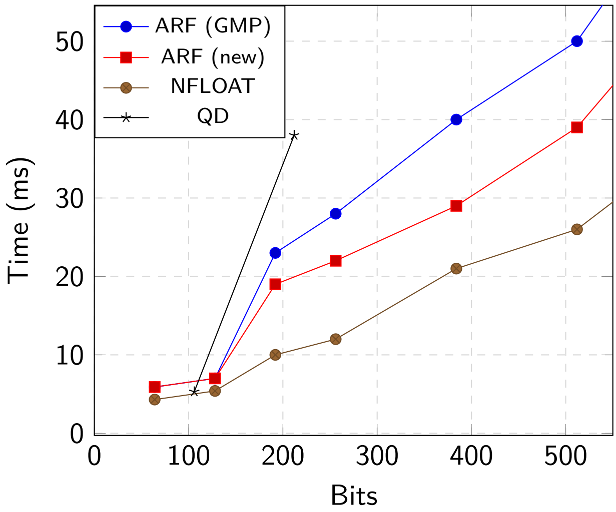


# Multiply two $100 \times 100$ FP matrices using dot products (Zen 3)



# Conclusions and thoughts

- Critical functions require hardware awareness!
- Straight line programs (SLPs) can be important to reduce overhead when going from native data types to multiple precision arithmetic
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# Bibliography


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