On Hadamard's Maximal Determinant Problem

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April 2009

AUSTRALIAN RESEARCH COUNCIL
Centre of Excellence for Mathematics
and Statistics of Complex Systems

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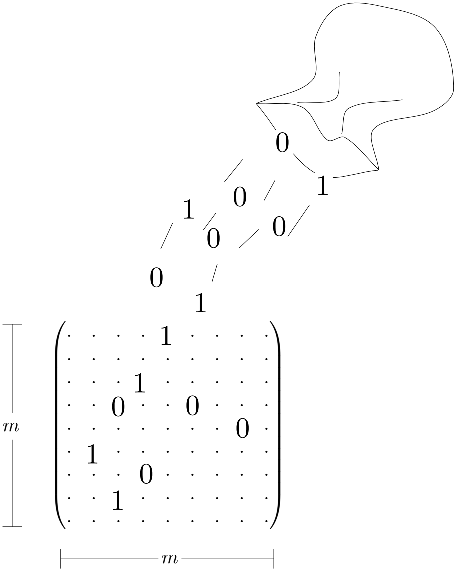

$$\max \det = ?$$

A Naive Computer Search

Order	max det	Time
1	1	*fast*
2	1	*fast*
3	2	*fast*
4	3	*fast*
5	5	*fast*
6	9	*order of days*
7	32	*order of years*
8	56	*order of the age of the Universe*

As well by hand? I found nested Max Dets ...

$$\begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & 1 \end{pmatrix} \color{red}{\}1}$$

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$$\begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 1 & 0 \\ & & & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ \color{red}\left.\begin{matrix} \\ \left.\begin{matrix} \\ \end{matrix}\right\}1 \end{matrix}\right\}1 \end{matrix}$$

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$$\begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & 1 & 0 & 1 \\ & & & 1 & 1 & 0 \\ & & & 0 & 1 & 1 \end{pmatrix} \left.\begin{matrix} \\ \left.\begin{matrix} \\ \left.\begin{matrix} \\ \end{matrix}\right\}1 \end{matrix}\right\}1 \end{matrix}\right\}2$$

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$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}
\begin{array}{l} \\ \\ \\ \\ \\ \left.\vphantom{1}\right\}1 \end{array}
\begin{array}{l} \\ \\ \\ \\ \left.\vphantom{\begin{matrix}1\\1\end{matrix}}\right\}1 \end{array}
\begin{array}{l} \\ \\ \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}2 \end{array}
\begin{array}{l} \\ \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}3 \end{array}
\begin{array}{l} \\ \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\}5 \end{array}
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- But no further!

The problem turns out to be famous

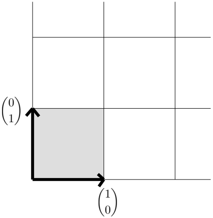
- Hadamard's Maximal Determinant Problem was posed in 1893

Jacques Hadamard

- A little selected history on this century-old question …

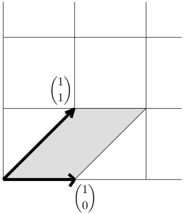
Observe:

- 2D:



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

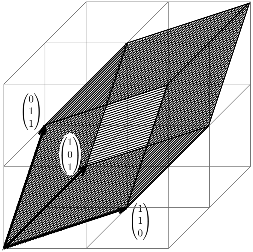
or



$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Geometry: max |det| = max (hyper-)Volume

- 3D:



$$\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}$$

An equivalent problem: $\{+1, -1\}$ matrices

$$\begin{tikzcd}
\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \arrow[r, "\times(-2)"] & \begin{pmatrix} -2 & 0 & -2 \\ -2 & -2 & 0 \\ 0 & -2 & -2 \end{pmatrix} \arrow[r, "\text{border}"] & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix} \arrow[r, "\text{add row 1}"] & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \arrow[d, "\text{column ops}"] \\
m \times m \text{ matrix} & & & \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \arrow[d, "\text{row ops}"] \\
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\end{tikzcd}$$

- $|\det_{\text{new}}| = 2^m |\det_{\text{old}}|$

Volume interpretation ⇒ upper bound on |max det|

$$\max \underbrace{\left|\begin{matrix} \pm1 & \cdots & \pm1 \\ \vdots & \ddots & \vdots \\ \pm1 & \cdots & \pm1 \end{matrix}\right|}_{n} \left.\vphantom{\begin{matrix} \pm1 \\ \vdots \\ \pm1 \end{matrix}}\right\} n$$

- What is the upper bound?

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- What is the upper bound?

$$\left(\sqrt{(\pm 1)^2 + \cdots + (\pm 1)^2}\right)^n = n^{n/2}$$

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- Why?

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- Why? (Columns/rows orthogonal)

When is the bound tight?

- Tight when $\{+1, -1\}$ square matrix H of order n satisfies

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- *Hadamard Conjecture (Paley, 1933):* this is also *sufficient*.

Evidence for Hadamard Conjecture

- Many constructions for infinite families, including
  - Sylvester, $\forall\ 2^r$
  - First Paley, using finite fields, $\forall\ p^r + 1$, p prime
  - Second Paley, using finite fields, $\forall\ 2p^r + 2$, p prime

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- Smallest $n \equiv 0 \pmod 4$ currently undecided:

$$n = 668.$$

More evidence

Order	Number of inequivalent Hadamard matrices – see Sloan's sequence A007299
1	1
2	1
4	1
8	1
12	1
16	5
20	3
24	60
28	487
32	$\geq 3\,578\,006$
36	$\geq 18\,292\,717$

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$n \equiv 1$	1	5	9	13	17	21	25
$\frac{\lvert\text{max det}\rvert}{2^{n-1}}$	1	3×1^1	7×2^3	15×3^5	20×4^7	29×5^9	42×6^{11}

The smallest unknown order is n=29.

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$n \equiv 2$	2	6	10	14	18
$\frac{\lvert \max \det \rvert}{2^{n-1}}$	1	5×1^1	18×2^3	39×3^5	68×4^7

  The smallest unknown order is n=22.

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The smallest unknown order is n=22.

$n \equiv 3$	3	7	11	15
$\frac{\lvert\max \det\rvert}{2^{n-1}}$	1	9×1^1	40×2^3	105×3^5

The smallest unknown order is n=19.

Tighter upper bounds?

- The *Hadamard bound* of $n^{n/2}$ holds for all orders but is never tight for $n \not\equiv 0 \pmod 4$ when $n > 2$.

Tighter upper bounds?

- The *Hadamard bound* of $n^{n/2}$ holds for all orders but is never tight for $n \not\equiv 0 \pmod 4$ when $n > 2$.
- Better upper bounds for non-Hadamard orders were proved by
    - Barba in 1933
    - Ehlich in 1962, 64
    - Wojtas in 1964
    - Cohn (proved a new bound tightness result) in 2000

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- The *Ehlich-Wojtas bound* holds for $n \equiv 2 \pmod 4$:

$$(2n-2)(n-2)^{(n/2)-1}$$

The best known upper bounds are:

- The *Ehlich bound* holds for $n \equiv 3 \pmod 4$:

$$(n-3)^{\frac{n-s}{2}}(n-3+4r)^{\frac{u}{2}}(n+1+4r)^{\frac{v}{2}}\sqrt{1-\frac{ur}{n-3+4r}-\frac{v(r+1)}{n+1+4r}}$$

where $s = 3$ for $n = 3$, $s = 5$ for $n = 7$, $s = 5$ or 6 for $n = 11$, $s = 6$ for $n = 15, 19, ..., 59$, and $s = 7$ for $n \geq 63$, $r = \lfloor \frac{n}{s} \rfloor$, $n = rs + v$ and $u = s - v$.

Percentages of bounds met: summary from Will Orrick's:

- www.indiana.edu/∼maxdet

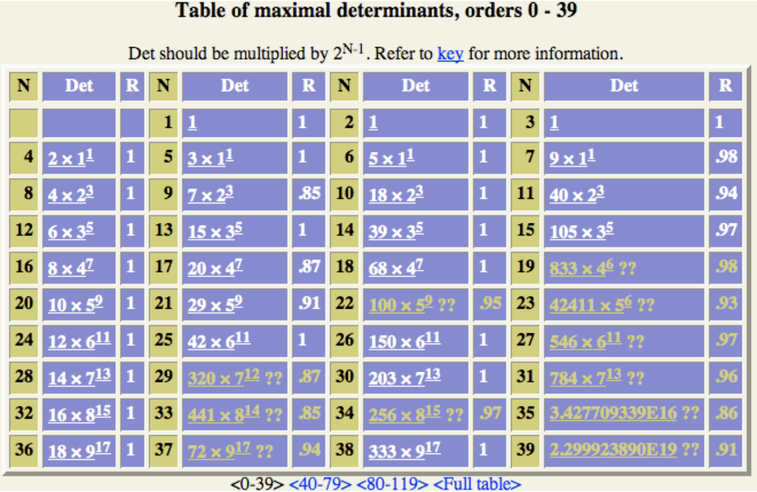

Table of maximal determinants, orders 0 - 39

Det should be multiplied by 2^{N-1}. Refer to key for more information.

N	Det	R	N	Det	R	N	Det	R	N	Det	R
			1	1	1	2	1	1	3	1	1
4	2×1^1	1	5	3×1^1	1	6	5×1^1	1	7	9×1^1	.98
8	4×2^3	1	9	7×2^3	.85	10	18×2^3	1	11	40×2^3	.94
12	6×3^5	1	13	15×3^5	1	14	39×3^5	1	15	105×3^5	.97
16	8×4^7	1	17	20×4^7	.87	18	68×4^7	1	19	833×4^6 ??	.98
20	10×5^9	1	21	29×5^9	.91	22	100×5^9 ??	.95	23	42411×5^6 ??	.93
24	12×6^{11}	1	25	42×6^{11}	1	26	150×6^{11}	1	27	546×6^{11} ??	.97
28	14×7^{13}	1	29	320×7^{12} ??	.87	30	203×7^{13}	1	31	784×7^{13} ??	.96
32	16×8^{15}	1	33	441×8^{14} ??	.85	34	256×8^{15} ??	.97	35	3.427709339E16 ??	.86
36	18×9^{17}	1	37	72×9^{17} ??	.94	38	333×9^{17}	1	39	2.299923890E19 ??	.91

<0-39> <40-79> <80-119> <Full table>

The idea behind the bounds

- Let R be a maximal determinant square ±1 matrix of order n.

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- Let $\mathcal{G}_n$ be the set of all gram matrices, G, of order n

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The idea behind the bounds, continued

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- Then
$$\overline{\mathcal{G}}_n \supseteq \mathcal{G}_n$$
- Hence
$$\left|\text{max det}\left(\overline{\mathcal{G}}_n\right)\right| \geq \left|\text{max det}\left(\mathcal{G}_n\right)\right|$$

Case: $n \equiv 1 \pmod 4$

- The matrix which was proven by Barba and Ehlich to have largest determinant in $\overline{\mathcal{G}}_n$ is

$$\begin{pmatrix} n & & 1 \\ & \ddots & \\ 1 & & n \end{pmatrix}$$

Case: $n \equiv 2 \pmod 4$

- The matrix which was proven by Ehlich and Wojtas to have largest determinant in $\overline{\mathcal{G}}_n$ is

$$\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}, \text{ where } F = \begin{pmatrix} n & & 2 \\ & \ddots & \\ 2 & & n \end{pmatrix}$$

Case: $n \equiv 3 \pmod 4$

- We expect a matrix with largest determinant in $\overline{\mathcal{G}}_n$ to be:

$$\begin{pmatrix} n & & -1 \\ & \ddots & \\ -1 & & n \end{pmatrix}$$

- In general, this is wrong
- Ehlich proved: the correct best determinant matrix in $\overline{\mathcal{G}}_n$ is a block form with off-diagonal entries from the set $\{-1, +3\}$.

Structure in the $n \equiv 1 \pmod 4$ case

- What is a necessary condition for tightness of: *Barba-Ehlich:*
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- What is a necessary condition for tightness of: *Barba-Ehlich:*

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- **Lemma:** The number $2n-1$ is a perfect square iff $\exists q \in \mathbb{N}$ such that

$$n = q^2 + (q+1)^2$$

In the literature, a conjecture on tightness:

- **Conjecture:** The Barba-Ehlich bound is tight whenever n is a sum of two consecutive squares:

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$$q = 2, 4$$

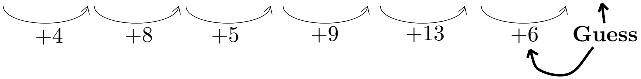
and when

$$q = p^r$$

for p an odd prime – proved by Brouwer's Construction.

What chance an exact maxdet formula ∀ $n \equiv 1 \pmod 4$?

$n \equiv 1$	**1**	**5**	9	**13**	17	21	**25**	29
$\frac{\lvert\text{max det}\rvert}{2^{n-1}}$	1	3×1^1	7×2^3	15×3^5	20×4^7	29×5^9	42×6^{11}	48×7^{13}



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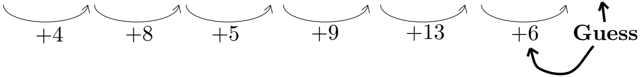
+4
+8
+5
+9
+13
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Guess

- A guess/conjecture due to Will Orrick is that $\frac{|\text{max det}|}{2^{n-1}}$ for $n = 4k + 1$ is always divisible by

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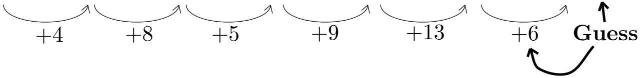

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with coefficients growing quadratically between n's for which $n = q^2 + (q+1)^2$.

What can we hope to compute?

- The ideas of Ehlich and Wojtas were reused by Chadjipantelis, Kounias and Moissiadis in the 1980's to find max det matrices for $n = 17$ and $n = 21$.

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- Are we within reach of $n = 29$?

Basic Idea

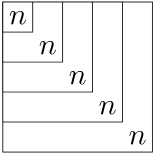
Two steps:

1. Find candidate gram matrices in $\overline{\mathcal{G}}_n$.
2. Check if candidates decompose in the form

$$RR^T.$$

Essentials of Step 1.

- There exist theorems which bound the determinants of candidate gram matrices in terms of their sub-matrices.
- So we can set a target determinant and build candidates:



pruning *too-small* sub-matrices as we go.

Computational considerations for Step 1.

- We must have **efficient** ways to calculate determinants!
- 'Rank-One Update' Theorems:

$$O(\text{size}^3) \rightarrow O(\text{size}^2),$$

at the expense of some book-keeping.

Computational considerations for Step 1.

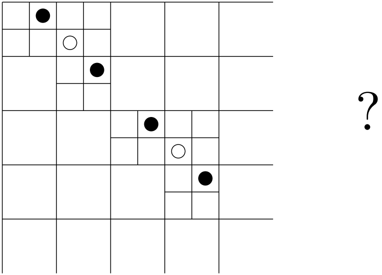
- Need to prune equivalent gram matrices:

eg. $$\begin{pmatrix} 7 & 3 & -1 & -1 & -1 & -1 & -1 \\ 3 & 7 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 7 & 3 & -1 & -1 & -1 \\ -1 & -1 & 3 & 7 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 7 & 3 & -1 \\ -1 & -1 & -1 & -1 & 3 & 7 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 7 \end{pmatrix} \sim \begin{pmatrix} 7 & -1 & -1 & 3 & -1 & -1 & -1 \\ -1 & 7 & 3 & -1 & -1 & -1 & -1 \\ -1 & 3 & 7 & -1 & -1 & -1 & -1 \\ 3 & -1 & -1 & 7 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 7 & 3 & -1 \\ -1 & -1 & -1 & -1 & 3 & 7 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 7 \end{pmatrix}$$

under simultaneous row and column permutation

Computational considerations for Step 1.

- So need a (partial ordering) which
  - prunes heavily enough, and
  - is practical to compute on-the-fly



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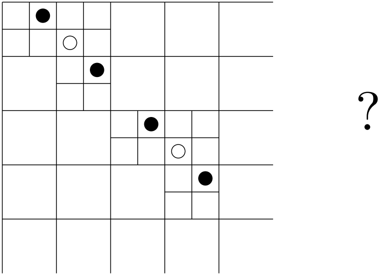

- Make sure we don't miss any valid candidates!

Essentials of Step 2.

- Consider gram candidates A and B whose determinants agree.

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$$RR^T \text{ and } R^T R$$

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- Implement two kinds of constraints:
  - Linear,
  - Quadratic – which use A and B simultaneously

Essentials of Step 2. – the linear constraints

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- We need to implement an ordering to prune duplicates

Considerations for the linear constraints of Step 2.

- eg.

$$A = \begin{pmatrix} 17 & -3 & 1 & \cdots \\ -3 & 17 & 1 & \cdots \\ 1 & 1 & 17 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

  If we have already built

  $\mathbf{r}_0 = (1,\ -,\ -,\ 1, 1,\ -, -,\ -, -, -,\ 1, 1, 1,\ 1, 1, 1, 1)$ and
  $\mathbf{r}_1 = (-,\ -,\ 1,\ -, -,\ -, -,\ 1, 1, 1,\ -, -, -,\ 1, 1, 1, 1)$

  then $\mathbf{r}_2$ breaks into blocks:

  $\mathbf{r}_2 = (\mathrm{a};\ \mathrm{b};\ \mathrm{c};\ \mathrm{d}, \mathrm{e};\ \mathrm{f}, \mathrm{g};\ \mathrm{h}, \mathrm{i}, \mathrm{j};\ \mathrm{k}, \mathrm{l}, \mathrm{m};\ \mathrm{n}, \mathrm{o}, \mathrm{p}, \mathrm{q})$

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- Adding more rows is a process of successive block refinement

Considerations for the linear constraints of Step 2.

- Because we work with both rows and columns, we need a way of refining row-blocks and column blocks simultaneously!

1	−	−	1	1	−	−	
−	−	1	−	−	−	−	
−	−						
1	1						
1	1						
1	−						

?

Essentials of Step 2. – the quadratic constraints

- To derive the quadratic constraints, write key ingredients in block form:

$$R = \begin{pmatrix} 1 & \mathbf{y^T} \\ \mathbf{x} & R' \end{pmatrix}, R^T = \begin{pmatrix} 1 & \mathbf{x^T} \\ \mathbf{y} & R'^T \end{pmatrix}, A = \begin{pmatrix} n & \mathbf{a^T} \\ \mathbf{a} & A' \end{pmatrix}, B = \begin{pmatrix} n & \mathbf{b^T} \\ \mathbf{b} & B' \end{pmatrix}$$

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- This allow several quadratic constraints to be found, eg.

$$\det(A' - \mathbf{x}\mathbf{x}^T) = \text{ a perfect square } = \det(B' - \mathbf{y}\mathbf{y}^T)$$

Computational Considerations for Step 2.

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- How to decide?

THE END