The probability that the number of points on the Jacobian of a genus two curve is prime

Wouter Castryck, Hendrik Hubrechts, Alessandra Rigato

Part I: the elliptic curve case

- Say we wish to generate an elliptic curve $E/\mathbb{F}_q$ suitable for use in discrete-log based cryptosystems.
- SPH attack $\rightsquigarrow \#E(\mathbb{F}_q)$ should have a large prime factor.
- Two approaches:
  - Fix n and construct $E/\mathbb{F}_q$ such that $\#E(\mathbb{F}_q) = n$ (using CM).
  - Fix q and try random $E/\mathbb{F}_q$ until $\#E(\mathbb{F}_q)$ has a large prime factor (using point counting algorithms).
- Central question: what is the probability of success?
- For simplicity, throughout this talk we will:
  - restrict to prime fields $\mathbb{F}_p$;
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- Aim of Part I: 'rediscover' a concrete conjecture due to Galbraith & McKee.

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- Let $\mathbb{F}_p$ be a finite prime field, $p > 3$.
- Let $E : y^2 = x^3 + Ax + B$ be a randomly chosen elliptic curve over $\mathbb{F}_p$.
  - That is: (A, B) is chosen from the finite set

$$\left\{ (A, B) \in \mathbb{F}_p^2 \;\middle|\; 4A^3 + 27B^2 \neq 0 \right\}$$

    uniformly at random.
- By Hasse's theorem, the number N_E of (projective) rational points on E is contained in

$$\left[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}\right].$$

- If N_E were uniformly distributed, we would expect

$$P(N_E \text{ is prime}) \approx \frac{1}{\log p}$$

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- For growing p, N_E tends to follow a semicircular distribution.
  - Translate to obtain

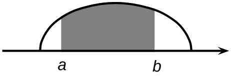
$$T_E = N_E - (p+1) \in [-2\sqrt{p}, 2\sqrt{p}]$$

    (trace of Frobenius).
  - Rescale to obtain

$$t_E = T_E / 2\sqrt{p} \in [-1, 1].$$

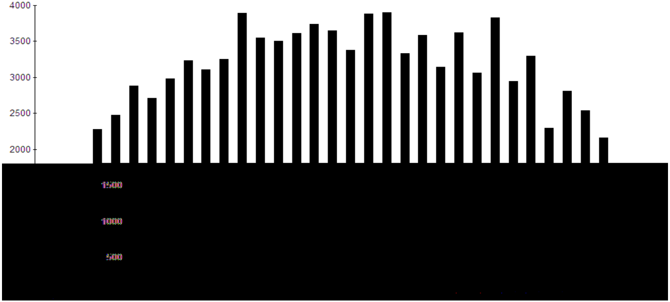
  - Then for any $a < b$ in $[-1, 1]$

$$\lim_{p\to\infty} P(a < t_E < b) = \int_a^b \frac{2}{\pi}\sqrt{1-t^2}\,dt.$$

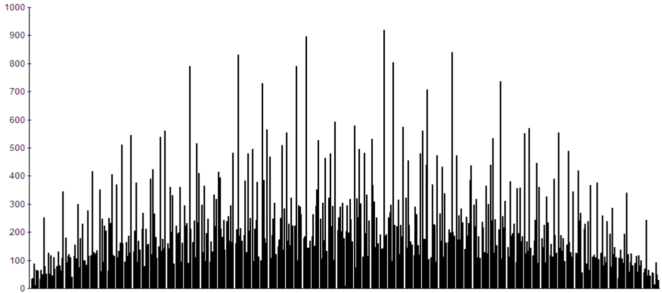


  - Proof of Birch uses theory of bivariate quadratic forms.

- Experimental evidence: a histogram of 100.000 curves $y^2 = x^3 + Ax + B$ over $\mathbb{F}_{7^5}$, with interval width 15:



- The limit hides some subtleties that are related to the discrete nature of N_E (or T_E).
- Same experiment, but now interval width 1:



- This doesn't seem to converge to a semicircle very 'smoothly' (lots of peaks and valleys).
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- Easy fact (not very well-known):

$$\lim_{p\to\infty} P(N_E \text{ is even}) = \frac{2}{3}.$$

- Proof:
  - The completing-the-cube map

    $$\{\text{square-free } x^3+ax^2+bx+c\} \to \{\text{square-free } x^3+Ax+B\}$$

    is uniform.
  - Thus we may assume that E is defined by $y^2 = f(x)$ for a random square-free $f(x) = x^3 + ax^2 + bx + c$.
  - N_E is even $\Leftrightarrow E(\mathbb{F}_p)$ has 2-torsion $\Leftrightarrow f(x)$ is reducible.
  - The *irreducible* $f(x)$ are precisely the minimal polynomials of all $\theta \in \mathbb{F}_{p^3} \setminus \mathbb{F}_p$ and the correspondence is 3-to-1.
  - Thus

    $$\lim_{q\to\infty} P(f(x) \text{ is irreducible}) = \lim_{p\to\infty} \frac{\frac{1}{3}(p^3-p)}{p^3 - O(p^2)} = \frac{1}{3} \qquad \blacksquare$$

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Let ℓ be any prime number, then

$$\lim_{p\to\infty}\left(P(\ell \mid N_E) - \begin{cases} \frac{1}{\ell-1} & \text{if } p \not\equiv 1 \bmod \ell \\ \frac{\ell}{\ell^2-1} & \text{if } p \equiv 1 \bmod \ell \end{cases}\right) = 0.$$

- Lenstra used this for estimating the complexity of his elliptic curve based integer factorization algorithm.
- Error term is $O(\ell/\sqrt{p})$.
- Note that in particular:

$$\ell \ll p \implies P(\ell \mid N_E) > \frac{1}{\ell},$$

so this suggests that $P(N_E \text{ is prime})$ is presumably smaller than one would naively expect.

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- Proof sketch in case $p \not\equiv 1 \bmod \ell$:
  - One clearly has
    $$\ell \mid N_E \iff E(\mathbb{F}_p) \text{ contains a point of order } \ell.$$
  - $p \not\equiv 1 \bmod \ell$ then implies that
    $$\ell \mid N_E \iff E(\mathbb{F}_p) \text{ contains exactly } \ell - 1 \text{ points of order } \ell.$$
  - These appear in $\frac{\ell-1}{2}$ pairs $\pm P$.
  - There exists a curve $X_1(\ell)/\mathbb{F}_p$ whose $\mathbb{F}_p$-rational points are in 1-1-correspondence with the set
    $$\{(E, \pm P) \mid E \text{ ell. curve } /\mathbb{F}_p,\ P \in E(\mathbb{F}_p) \text{ has order } \ell\}$$
    (e.g. defined by $\psi_\ell(E_j)(x) \in \mathbb{F}_p(j, x)$).
  - Therefore,
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- If $\ell \nmid p-1$ then

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$$P(\ell \nmid N_E) \approx \frac{\ell^2 - \ell - 1}{\ell^2 - 1} \quad \text{vs.} \quad P(\ell \mid \text{random number}) \approx \frac{\ell - 1}{\ell}.$$

- Let $P_1(p)$ be the probability that a random number from the Hasse interval is prime.
- Let $P_2(p) = P(N_E \text{ is prime})$.
- Heuristically,

$$P_1(p) \approx \prod_{\ell \leq \sqrt{p}} \frac{\ell - 1}{\ell} \approx \frac{p}{\log p}.$$

- Heuristically (using Lenstra's estimates),

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Conjecture (Galbraith-McKee)

Let

$$c_p = \frac{2}{3} \cdot \prod_{\ell > 2} \left(1 - \frac{1}{(\ell-1)^2}\right) \cdot \prod_{\ell | p-1,\ \ell > 2} \left(1 + \frac{1}{(\ell+1)(\ell-2)}\right),$$

then

$$\lim_{p \to \infty} \left(P_2(p)/P_1(p) - c_p\right) = 0.$$

- $c_p \in [0.44, 0.62]$
- Galbraith & McKee give different heuristics!
- They use the analytic Hurwitz-Kronecker class number formula

$$H(t^2 - 4p) = \frac{\sqrt{4p - t^2}}{\pi} \cdot \prod_{\ell} \left\{ \left(1 - \left(\frac{t^2 - 4p}{\ell}\right)/\ell\right)^{-1} \psi_3(\ell) \right\}$$

counting equivalence classes of bivariate quadratic forms.

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- If one follows our heuristics to estimate the number of elliptic curves with given trace t, one obtains

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- E.g. for $\ell = 2$, the correcting factor is
  - $2/3$ if t is odd,
  - $4/3$ if t is even.
- This turns out to be a reformulation of the analytic Hurwitz-Kronecker class number formula!
- Schematically:

Lenstra	$\hookrightarrow$	Galbraith-McKee	$\hookrightarrow$	Hurwitz-Kronecker
$P(\ell \mid N_E)$		$P(N_E \text{ prime})$		$P(N_E = n)$
proven (algebraic)		conjectural		proven (analytic)
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The random matrix model

- Let $\gcd(n,p) = 1$. To an elliptic curve $E/\mathbb{F}_p$ we can associate its n-torsion subgroup
$$E[n] = \{P \in E(\overline{\mathbb{F}}_p) \mid [n]P = \infty\}.$$
It is well-known that
$$E[n] \cong \mathbb{Z}/(n) \times \mathbb{Z}/(n).$$

- Let (P,Q) be a $\mathbb{Z}/(n)$-module basis of $E[n]$, and let $\sigma : E[n] \to E[n]$ be pth power Frobenius. Then we can write
$$P^\sigma = [\alpha]P + [\beta]Q, \quad Q^\sigma = [\gamma]P + [\delta]Q.$$

- Important fact: the matrix
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in (\mathbb{Z}/(n))^{2\times 2}$$
has trace $\equiv T_E \bmod n$ and determinant $\equiv p \bmod n$.

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- A different choice of basis results in a $\mathrm{GL}_2(\mathbb{Z}/(n))$-conjugated matrix.
- Thus we can unambiguously associate to E a *conjugacy class* $\mathcal{F}_E$ of matrices of Frobenius (all having trace T_E and determinant p).
- Let $\mathcal{M}_p \subset \mathrm{GL}_2(\mathbb{Z}/(n))$ be the set of *all* matrices of determinant p.

Theorem (Katz-Sarnak, Achter, C.-Hubrechts)

Let $\mathcal{F}$ be a conjugacy class of matrices of determinant p. Then

$$\lim_{p\to\infty}\left(P(\mathcal{F}_E = \mathcal{F}) - \frac{\#\mathcal{F}}{\#\mathcal{M}_p}\right) = 0.$$

The error term is $Cn^2/\sqrt{p}$, where C is an explicit and absolute constant.

- Idea of proof: apply Chebotarev's density theorem to the modular covering $X(n) \to X(1)$.

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- Example to get in touch with the flavor:
- What proportion of elliptic curves satisfies $E[\ell] \subset E(\mathbb{F}_p)$?
  - $E[\ell] \subset E(\mathbb{F}_p)$ if and only if $E[\ell]$ has a basis consisting of $\mathbb{F}_p$-rational points P and Q.
  - Thus: if and only if

$$\mathcal{F}_E = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

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Part II: the genus 2 case

- Say we wish to generate a genus 2 hyperelliptic curve $H/\mathbb{F}_q$ suitable for use in discrete-log based cryptosystems.
- SPH attack $\rightsquigarrow \#\mathbb{J}(H)(\mathbb{F}_q)$ should have a large prime factor.
- Two approaches:
  - Fix n and construct $H/\mathbb{F}_q$ such that $\#\mathbb{J}(H)(\mathbb{F}_q) = n$ (using CM).
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- Again: what is the probability of success?
- For simplicity, we will:
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- Aim of Part II: generalize the Galbraith-McKee conjecture.

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- Let $\mathbb{F}_p$ be a finite prime field, $p > 2$.
- Let $H : y^2 = f(x)$ be a randomly chosen genus 2 curve over $\mathbb{F}_p$. That is:
  - Either $f(x)$ is chosen from the finite set
  $$\{f(x) \in \mathbb{F}_p[x] \mid f(x) \text{ monic and squarefree, } \deg f(x) = 5\}$$
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  uniformly at random.
- By Tate's theorem, the number N_H of rational points on $\mathbb{J}(H)$ is contained in
$$[(p+1)^2 - 4(p+\sqrt{p}+1)\sqrt{p}, (p+1)^2 + 4(p+\sqrt{p}+1)\sqrt{p}].$$
- First remark: the above notions are fundamentally different!

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Lemma

Let $H/\mathbb{F}_q$ be a curve of genus 2. Each of the 15 non-trivial 2-torsion points of $\mathbb{J}(H)$ (thought of as a divisor class) contains a unique pair of divisors $\{P_i - P_j, P_j - P_i\}$, where P_i and P_j are distinct Weierstrass points.

- Proof:
- Think of the P_i as the points $(x_i, 0)$ on some Weierstrass model $y^2 = f(x)$ with $\deg f = 6$.
- $2P_i - 2P_j \sim 0$, hence $P_i - P_j \sim P_j - P_i$ has 2-torsion.
- $P_i - P_j \not\sim 0$ by Riemann-Roch.
- All pairs are distinct:
  - $(P_1 - P_2) - (P_1 - P_3) \sim P_2 - P_3$.
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- There are $\binom{6}{2} = 15$ such point pairs, so every 2-torsion points must appear in this way. ∎

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- Corollary: $\mathbb{J}(H)$ has 2-torsion if and only if H has an '$\mathbb{F}_p$-rational' Weierstrass point pair:
  - either two $\mathbb{F}_p$-rational Weierstrass points,
  - either two Weierstrass points that are swapped by Galois conjugation.
- In degree 6:
  - $f(x)$ has two linear factors or
  - $f(x)$ has a quadratic factor.

  Exercise: probability that this happens is $\approx 26/45 \approx 58\%$.
- In degree 5, our curve automatically has an $\mathbb{F}_p$-rational Weierstrass point. Thus there is 2-torsion if
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  or in other words if $f(x)$ is reducible! The probability that this happens is $\approx 4/5 = 80\%$ (same proof as before).
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The random matrix model in genus 2

- Let $\gcd(n,p) = 1$. To a genus 2 curve $H/\mathbb{F}_p$ we can associate the n-torsion subgroup of $\mathbb{J}(H)$
$$\mathbb{J}(H)[n] = \left\{ P \in \mathbb{J}(H)\left(\overline{\mathbb{F}}_p\right) \mid [n]P = O \right\}.$$
It is well-known that
$$E[n] \cong \mathbb{Z}/(n) \times \mathbb{Z}/(n) \times \mathbb{Z}/(n) \times \mathbb{Z}/(n).$$

- Let (P,Q,R,S) be a $\mathbb{Z}/(n)$-module basis of $\mathbb{J}(H)[n]$, and let $\sigma : \mathbb{J}(H)[n] \to \mathbb{J}(H)[n]$ be pth power Frobenius. Then we can write
$$P^\sigma = [\alpha]P + [\beta]Q + [\gamma]R + [\delta]S, \ldots.$$

- Important fact: the corresponding matrix in
$$(\mathbb{Z}/(n))^{4\times 4}$$
has trace $\equiv T_E \bmod n$ and determinant $\equiv p \bmod n$.

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- However: we will no longer consider *any* basis!
- $\mathbb{J}(H)$ is endowed with a *symplectic structure*, induced by the Weil pairing. We will restrict to symplectic bases.
- Now we associate to H an orbit under $\mathrm{GSp}_4(\mathbb{Z}/(n))$-conjugation of matrices in $\mathrm{Sp}_4^{(p)}(\mathbb{Z}/(n))$. Denote this orbit by $\mathcal{F}_H$.

Theorem (Katz-Sarnak, Achter, work to be done)

Let $H : y^2 = f(x)$ be a genus 2 curve, where $f(x)$ is chosen from

$$\{f(x) \in \mathbb{F}_p[x] \mid f(x) \text{ squarefree, } \deg f(x) = 6\}$$

uniformly at random. Let $\mathcal{F}$ be an orbit under $GSp_4(\mathbb{Z}/(n))$-conjugation. Then

$$\lim_{p\to\infty}\left(P(\mathcal{F}_H = \mathcal{F}) - \frac{\#\mathcal{F}}{\#Sp_4^{(p)}(\mathbb{Z}/(n))}\right) = 0.$$

- So: counting appropriate matrices!
- $\#\mathrm{Sp}_4(\mathbb{Z}/(\ell)) = \#\mathrm{Sp}_4^{(p)}(\mathbb{Z}/(\ell)) = \ell^4(\ell^4 - 1)(\ell^2 - 1)$
- We guess (via interpolation) that the proportion of $M \in \mathrm{Sp}_4^{(p)}(\mathbb{Z}/(\ell))$ for which $\chi(M)(1) = 0$ equals

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- Let $P_1(p)$ be the probability that a random number from the Weil interval is prime.
- Let $P_2(p) = P(N_H \text{ is prime})$.

Conjecture

Let

$$c_p = \frac{38}{45} \prod_{\ell > 2} \left(1 - \frac{1}{(\ell-1)^2} + \frac{\ell}{(\ell-1)^2(\ell^2-1)}\right) \prod_{\ell \mid p-1, \ell > 2} \left(1 + \frac{\ell^4 - \ell^3 - \ell - 2}{(\ell+1)(\ell^2+1)(\ell^3 - 2\ell^2 - \ell + 3)}\right),$$

then

$$\lim_{p \to \infty} \left(P_2(p)/P_1(p) - c_p\right) = 0.$$

- $c_p \in [0.63, 0.80]$ (cf. elliptic curves: $[0.44, 0.62]$)
- Recall: the above is for random squarefree $f(x)$ of degree 6.
- For random squarefree monic $f(x)$ of degree 5, the factor $38/45$ must be replaced by $2/5$.
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